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Load case characterisation for gradient based optimisation with an aircraft global finite element model

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Abstract

Due to the dynamically-varying nature of aircraft loading scenarios, a large volume of global load cases are generated and required to be analysed in the Global Finite Element Model. One method to reduce the number of load cases is to use Singular Value Decomposition (SVD) to derive a smaller set of characteristic distributions which represents all the global load cases. The analysis result for this set of characteristic loads can be superimposed to create the internal load distributions for all the original load cases, with acceptable accuracy. In the structural optimisation process the load distributions change as the local components are optimised, so it is useful to calculate the sensitivities of local component design variables to the local load distribution. This paper proposes a variant of the sensitivity calculation process, which is appropriate for large scale gradient based optimisation. By using the SVD of the set of load cases, the number of sensitivity calculations, can be significantly reduced.

Nomenclature

Abbreviation

<i>DOF</i>	Degrees Of Freedom
<i>GFEM</i>	Global Finite Element Model
<i>LC</i>	Load case
<i>RF</i>	Reserve Factor
<i>SE</i>	Structural Element
<i>SMT</i>	Shear, Moment and Torque
<i>SVD</i>	Singular Value Decomposition

Roman Symbols

<i>c</i>	constant value
<i>l</i>	characteristic load vector
<i>n</i>	number of calculations, components, etc.
<i>u</i>	left eigenvector
<i>v</i>	right eigenvector

<i>x</i>	design variable
<i>A</i>	full load case matrix
<i>C</i>	local stiffness matrix
<i>D</i>	displacement matrix
<i>F</i>	external force matrix
<i>K</i>	global stiffness matrix
<i>L</i>	characteristic load matrix
<i>M</i>	linear relationship matrix
<i>N</i>	internal load matrix
<i>S</i>	scaling matrix
<i>U</i>	left eigenvector matrix
<i>V</i>	right eigenvector matrix

Greek Symbols

δ	approximation error
ϵ	relative approximation error
σ	singular values
Σ	singular values matrix

Subscripts and Superscripts

<i>k</i>	reduced rank of the matrix
<i>r</i>	full rank of the matrix
<i>T</i>	transpose of the matrix

1. Introduction

During the preliminary design stage, an aircraft global finite element model (GFEM) is optimised to obtain the minimum structural weight. The optimisation is done by changing the design parameters whilst satisfying all the design constraints, which are usually in terms of reserve factors. In a large airframe model there may be over 10^5 degrees of freedom, and thousands of design parameters are involved during the optimisation process.

The number of design parameters has become even larger since composite materials have become a major material for airframe design. These have additional design parameters, such as ply thickness and angle, which

can be optimised simultaneously with the geometric design parameters [1]. As a result, the size of the actual problem being optimised in the industry these days could be very large.

One of the difficulties is that large number of load cases needed to be evaluated to ensure that the design meets the target performance in different operational scenarios. Changing one of these design parameters can alter the load paths through the entire structure, so it is vital to optimise all the design parameters at the same time.

In the current aircraft load and structural design process, load cases are normally provided by the loads department in terms of an array of data. The array size could be very large due to several factors. Firstly, the structural motion of an aircraft and the external perturbations are dynamic, which means varying over a specified time domain. Secondly, the conditions themselves that the aircraft has to encounter vary enormously [2]. Load cases are generated at every specified time instance to replicate the dynamic motion at those operating conditions, so there are great deal of information. However, some of them might be repeated or not even significant.

One of the load case reduction techniques was employed in the optimisation process [3]. Load cases were filtered using the max-min envelope approach, which selects only the extreme loads near the periphery of the envelope. A significant number of load cases was eliminated. As the down-selection is entirely based on the load values and not on the actual structural response of the loads themselves, some of the neglected load cases that lie well inside the envelope may cause the structural failure.

The recent research at Queen's University Belfast [4], [5] showed that the singular value decomposition (SVD) can create a small set of characteristic loads that represents the behaviour of the entire load case set. The characteristic loads can be linearly combined to construct the matrix of the original load cases effectively. This paper will show how such an approach can be implemented into the optimisation process and contributes to significant computational saving.

2. Singular Value Decomposition

SVD is an elegant way to factorise a general rectangular matrix into a product of simpler matrices. Suppose A is a real m by n matrix ($m > n$) rank r ($r \leq n$). The SVD of A is generally expressed as

$$A = U \Sigma V^T \quad (1)$$

The products obtained from the SVD comprise of

two orthogonal matrices U and V and a diagonal matrix Σ .

The Σ matrix is a diagonal matrix in which the diagonal entries are the singular values σ arranged in order of magnitude. The matrices U and V , sometimes called left and right eigenvector matrices, are formulated from a set of orthonormal vectors i.e. $\mathbf{u}_1, \dots, \mathbf{u}_m$ and $\mathbf{v}_1^T, \dots, \mathbf{v}_n^T$. The \mathbf{u} and \mathbf{v} are in \mathbb{R}^n and \mathbb{R}^m , which are the column and row space of the A matrix respectively. Equation (1) can be alternatively written in the form of vectors as

$A =$

$$[\mathbf{u}_1 \dots \mathbf{u}_r \mid \mathbf{u}_{r+1} \dots \mathbf{u}_m] \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ \hline & & & 0 & & \\ 0 & & & & 0 & \end{array} \right] \left[\begin{array}{c} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \\ \hline \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{array} \right] \quad (2)$$

All the vectors are linearly independent, so they form the orthonormal bases of the row and column spaces. Therefore, the SVD essentially classifies the bases of the row and column of the original matrix into two separate matrices and prioritises their magnitudes by the singular value matrix.

Additionally, there are only r singular values according to the rank of the original matrix. The singular values from $\sigma_{r+1}, \dots, \sigma_m$ are all zero. As a result, the vectors $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ and $\mathbf{v}_{r+1}^T, \dots, \mathbf{v}_n^T$ do not contribute any meaning to the matrix A . Only r components are required from each matrix to construct the original A matrix.

2.1. Reduced-rank approximation

One of the most important applications of the SVD is the reduced-rank approximation. As previously shown in equation (2), the SVD requires only r components to construct the original matrix A and fully preserves all of its information.

Since the singular values in diagonal of Σ are arranged in decreasing order, the last few singular values could be very small. Let the SVD of A be

$A =$

$$[\mathbf{u}_1 \dots \mathbf{u}_k \mid \mathbf{u}_{k+1} \dots \mathbf{u}_r] \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ \hline & & & \sigma_{k+1} & & \\ 0 & & & & \ddots & \\ & & & & & \sigma_r \end{array} \right] \left[\begin{array}{c} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_k^T \\ \hline \mathbf{v}_{k+1}^T \\ \vdots \\ \mathbf{v}_r^T \end{array} \right] \quad (3)$$

If $\sigma_k \gg \sigma_{k+1}$, the approximation of A is given by

$$A \approx A_k = U_k \Sigma_k V_k^T \quad (4)$$

such that A_k is an approximation of the original matrix A obtained from a reduced dimension of SVD matrices: U_k, Σ_k and V_k^T . Equation (4) is usually called the reduced-rank approximation or the rank- k approximation. This idea was initially given by Eckart and Young [6] and is very useful when there are only few significant singular values in the matrix A . The matrix A can be approximated from a few singular values along with a smaller set of U_k and V_k .

2.2. Error Quantification

Error arising from the reduced-rank approximation is unavoidable; however, it could be restricted by using a sufficient number of singular values. A method used to quantify the approximating error can be derived based on the Frobenius norm, which is the square-root of the sum of the squares of all elements in the matrix.

Using the fact that the Frobenius norm is invariant under orthogonal transformation [7], the matrices that consists of a set of orthogonal vectors i.e. U, V or U_k, V_k^T are all invariant under the Frobenius norm. Therefore, the Frobenius norm of A can be described by

$$\|A\|_F = \|U \Sigma V^T\|_F = \|\Sigma\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2} \quad (5)$$

Equation (5) implies that the Frobenius norm of A can be determined by the singular values alone. Similarly, the error from the reduced-rank approximation only depends on the ignored singular values. Denote $r - k$ as the partitioning starting from k^{th} rank to r^{th} rank; the approximation error is neatly defined by

$$\|\varepsilon\|_F = \|\Sigma_{r-k}\|_F = \sqrt{\sigma_{k+1}^2 + \dots + \sigma_r^2} \quad (6)$$

Finally, the relative error between the reduced-rank approximation and the original matrix can be fully described by

$$\delta = \frac{\|\varepsilon\|_F}{\|A\|_F} = \frac{\|\Sigma_{r-k}\|_F}{\|\Sigma\|_F} = \sqrt{\frac{\sigma_{k+1}^2 + \dots + \sigma_r^2}{\sigma_1^2 + \dots + \sigma_r^2}} \quad (7)$$

3. Characteristic loads

In the situation where an array of load cases provided for the analysis is very large, the SVD can be very useful. The main reason is that the loading patterns decomposed by the SVD are classified into the most fundamental forms. The rank- k approximation can eliminate unimportant information from the original data. If the array is arranged appropriately, the SVD can greatly reduce the redundancy of the data in the array.

3.1. Formulation of the characteristic loads

Let an array of external loads (forces or moments) in the d^{th} degree of freedom (DOF) be

$$a_d = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1j} \\ f_{12} & f_{22} & \dots & f_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i1} & f_{i2} & \dots & f_{ij} \end{bmatrix} \quad (8)$$

such that i is the number of load cases, and j is the number of nodes where the loads are applied. For a model with k DOFs, a matrix A consisting of k DOF loads can be written as

$$A = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_k \end{array} \right] \quad (9)$$

For example, a wing model which has 100 load cases with 3DOF (Shear, Moment and Torque) applied at 20 stations along the wing span will have 100 rows and 60 columns. The number of rows in A is the same as that in a . The number of its columns is expanded by the number of DOF. Hence, the total size of A is determined from

$$\text{size}(A) = m \times n = i \times (j \times k) \quad (10)$$

Performing the SVD on A will obtain 3 matrices: U, S and V^T . The basis vectors of the row and column of the matrix are categorised into the U and V respectively. Each of them is amplified or shrunk by the corresponding singular values (σ).

In order to derive a set of data that represents the characteristics of the loads, simply combine Σ and V^T by

$$L = \Sigma V^T \quad (11)$$

This can be presented in terms of vectors as

$$L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_r \end{bmatrix} = \begin{bmatrix} \sigma_1 v_1^T \\ \vdots \\ \sigma_r v_r^T \end{bmatrix} \quad (12)$$

Equation (12) shows that all the vectors ℓ_1, \dots, ℓ_r representing all the bases of the load space are arranged in order of magnitude. The matrix L is called the *Characteristic load* matrix as it contains all the basis vectors that characterise the loading behaviours (v^T) and their corresponding magnitudes (σ).

Considering the SVD of A in equation (1), it can alternatively be written in terms of characteristic loads as

$$A = UL \quad (13)$$

Applying the approximation theorem in equation (4) equation (13) can be expressed by k number of ℓ vectors by

$$A \approx A_k = U_k L_k \quad (14)$$

such that

$$L_k = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_k \end{bmatrix} = \begin{bmatrix} \sigma_1 \mathbf{v}_1^T \\ \vdots \\ \sigma_r \mathbf{v}_k^T \end{bmatrix} \quad (15)$$

Although at this stage, the definition of characteristic load appears to be just regrouping matrices into a newly defined variable, the usefulness of the concept will become more apparent during the analysis phase.

3.2. Balancing the magnitudes of the loading matrix

Preprocessing a matrix by balancing the relative magnitude of its components improves the result of an eigen value problem if it is poorly scaled [8]. This approach is indeed applicable for computation of the SVD.

Consider the loading matrix A given in equation (9). The loadings are a combination between forces and moments, their values and units are hence comparatively different. Additionally, the loading at each station can be relatively different i.e. shear forces near the wing root are generally higher than those near the tip. The key idea here is attempting to make the values in each column balanced relative to the others before performing the SVD. Consequently, the reference value, which should be obtained from every entry in each of the column vectors in A , is required.

Recall that the Euclidean norm of a vector in \mathbb{R}^p is the total length of vector in p dimensional space. Therefore, it can effectively be used as the reference value. Introduce a diagonal matrix S where its diagonal entry is the inverse norm of the corresponding column vector of A as

$$S = \begin{bmatrix} \|\mathbf{a}_1\|^{-1} & & & \\ & \|\mathbf{a}_2\|^{-1} & & \\ & & \ddots & \\ & & & \|\mathbf{a}_n\|^{-1} \end{bmatrix} \quad (16)$$

such that \mathbf{a}_n is a column vector in A . Consequently, a balanced matrix A' of any given loading matrix A is computed by

$$A' = AS \quad (17)$$

Notice that each column vector in A is multiplied by the corresponding diagonal entry in S . As a result, all the column vectors in A' are balanced by their relative

magnitude. Performing the SVD on this matrix results in

$$A' = U' \Sigma' V'^T \quad (18)$$

To convert the loading matrix back to its original scale, the matrix A' in equation (17) can simply be multiplied by the inverse of S as

$$A = A' S^{-1} \quad (19)$$

Substitute equation (18) into (19) gives an alternative SVD formulation, which can be written in terms of characteristic loads by

$$A = U' \Sigma' V'^T S^{-1} = U' L' S^{-1} \quad (20)$$

Equation (20) clearly indicates that all the information in the matrix A is fully preserved. Referring to equations 14, the SVD of A can be expressed by using rank k approximation by

$$A \approx A_k = U'_k \Sigma'_k V'^T_k S^{-1} = U'_k L'_k S^{-1} \quad (21)$$

Figure 1 and 2 illustrate the magnitudes (absolute) of the loads from the unbalanced and balanced matrix.

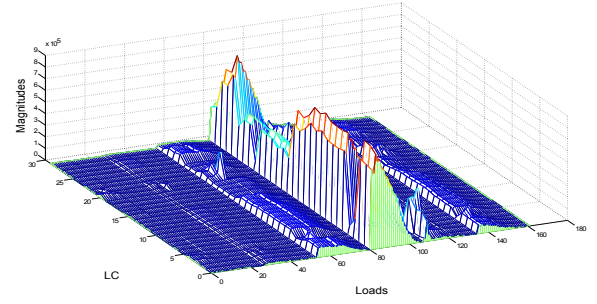


Figure 1: Original magnitudes of SMT loadings

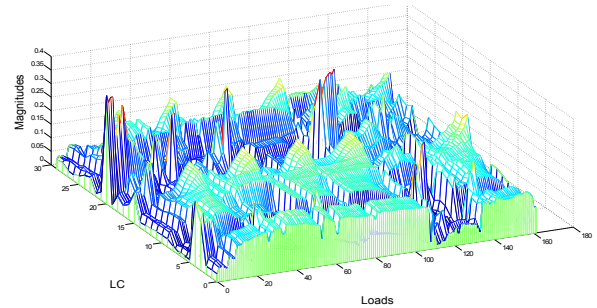


Figure 2: Balanced magnitudes of SMT loadings

4. Analysis of characteristic loads

In the aircraft structural design process, a GFEM is evaluated at each external load case to attain internal

load paths going into each sub-structure. The quantity of external load cases therefore directly indicates the number of analyses. When the analysis is performed linearly, incorporating the SVD into the process is possible and could significantly reduce the computational effort.

4.1. Linear analysis of characteristic loads

Suppose an output matrix of some linear responses such as internal loads (N) is requested from a GFEM analysis. N is typically related to the external force matrix (F) by a single matrix (M) that contains only material and geometric properties of the structure as

$$N = MF \quad (22)$$

To integrate the SVD into the finite element analysis, the A matrix of loads must be used instead of F . The format of F in finite element analysis is slightly different from the SVD format (A) shown in equation (10). Its rows and columns are generally expressed by

$$\text{size}(F) = n \times m = (k \times j) \times i \quad (23)$$

Notice that the size of the matrix F is essentially A^T . The only required operation is repositioning the rows in A^T from $(j \times k)$ to $(k \times j)$, which can be achieved by utilising a permutation matrix (P). This is done to ensure that rearranging the matrix will not cause any incompatibility in the multiplication process.

The permutation matrix is a square matrix whose rows and columns contain a single element with the value “1” and “0” elsewhere and normally used to reshape the current matrix into a desired pattern [9]. The relationship between the F and A^T can be defined as

$$F = PA^T \quad (24)$$

Both of A and F formats are fixed and governed by the same set of variables. Therefore, the pattern associating between the two matrices can be found and easily be programmed. Once the perturbation matrix is constructed, the formats of the A and F are interchangeable. Equation (22) is rewritten as

$$N = MPA^T \quad (25)$$

4.2. Superposition of characteristic loads

Since A can be decomposed by the SVD into characteristic load L and the corresponding U matrices, equation (25) can be replaced by

$$N = MPL^T U^T \quad (26)$$

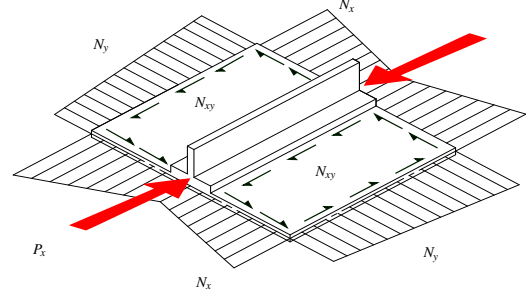


Figure 3: The in-plane loadings governed in a structural element

This equation is denoted as the *superposition of characteristic loads*, which is applicable for any finite element applications where the models and the requested outputs are linearly related.

The explanation of this equation is that L^T is rearranged to the new format and then analysed in finite element analysis. The result is the product of MPL^T . The multiplication between MPL^T and U^T produces the similar result with analysing A^T in equation (25).

Nevertheless, equation (26) still requires the analysis result of at least r characteristic loads, which does not exploit the full advantage of the SVD. Instead, the reduced rank approximation in equation (20) can be employed in order to further reduce the computational effort. This can be written as

$$N \approx N_k = MPL_k^T U_k^T \quad (27)$$

Equation (27) shows that all internal load results (N^T) analysed from the full load cases in A can be approximated from the analysis of only k load cases multiplied with the corresponding matrix product.

4.3. Error from the superposition of the k characteristic loads

In theory as long as all the characteristic loads are included, the superposition should give exactly the same outcome as analysing the full load cases. However, the error will be unavoidable when a fewer number of characteristic loads is used.

The ideal scenario is being able to evaluate the approximation error without running the full analysis. Recall that N is the internal load matrix due to the direct analysis of A . The error occurring from the superposition of k characteristic loads to determine N_k is given by

$$\varepsilon = \|N - N_k\|_F \quad (28)$$

Substituting equation (26) and (27) into (28) gives

$$\varepsilon = \|MPL^T U^T - MPL_k^T U_k^T\|_F \quad (29)$$

Since U and U_k are orthogonal invariant, then

$$\varepsilon = \|MP(L^T - L_k^T)\|_F = \|MPL_{r-k}^T\|_F \quad (30)$$

As a result, the relative error is equal to

$$\delta = \frac{\|\varepsilon\|_F}{\|A\|_F} = \frac{\|MPL_{r-k}^T\|_F}{\|MPL^T\|_F} \quad (31)$$

Equation (31) involves two characteristic loads terms: L , L_k and some constant matrices: M , P . From two matrix properties namely submultiplicative ($\|xy\|_F \leq \|x\|_F \|y\|_F$) [10] and orthogonal invariant, this equation can be rewritten as

$$\begin{aligned} \delta &\leq \frac{\|MP\|_F \|L_{r-k}^T\|_F}{\|MP\|_F \|L^T\|_F} = \frac{\|\Sigma_{r-k}\|_F}{\|\Sigma\|_F} \\ &\leq \sqrt{\frac{\sigma_{k+1}^2 + \dots + \sigma_r^2}{\sigma_1^2 + \dots + \sigma_r^2}} \end{aligned} \quad (32)$$

The approximation error at the internal load level is essentially defined by the magnitude of singular values. This equation implies that it should not be greater than and possess a similar trend with the error at the external load level in equation (7).

5. SVD based sensitivity analysis

Among the numerical procedures available today, gradient based optimisation is one of the most powerful techniques capable of handling a large-scale structural optimisation problem involving multiple design parameters, load cases and constraints. This paper presents a technique that could be suitable for efficient computing in numerical optimisation applications, especially in a large-scale gradient based optimisation.

5.1. Sensitivity of the reserve factor constraints with respect to the design variables

In gradient based optimisation process, sensitivities of the performance or the constraints, which define the most efficient searching directions toward the optimum, are required by the optimiser. The sensitivities are the changes in performance or constraint functions with respect to the changes in design parameter. Generally the constraints functions are defined in the form of reserve factors (RF) e.g. $RF > 1$, in order to ensure that the structure will be able to withstand the loads.

The RF for a given structure is a function of both design parameters (x) and the internal loads (N); the full

sensitivity equation must be expressed by the chain rule of differentiation as

$$\frac{dRF(x, N)}{dx} = \frac{\partial RF(x, N)}{\partial x} + \frac{\partial RF(x, N)}{\partial N} \cdot \frac{dN(x)}{dx} \quad (33)$$

Typically, calculating the sensitivities is one of the most expensive parts of the gradient based optimisation process because of a multitude of constraints, which are subjected to many loads and design parameters. Figure 4 shows an example of some typical design variables in a stringer-panel structural element.

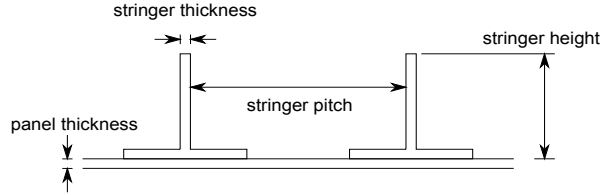


Figure 4: Typical design variables in a stringer-panel structure

The calculation process is done separately by two main steps. The first step involves determining the two partial derivatives: $\partial RF(x, N)/\partial x$ and $\partial RF(x, N)/\partial N$. These two terms are generally calculated via the finite differencing, which is computing the difference between the perturbed reserve factor and the existing reserve factor over a small change in design variables or internal forces. Most of the reserve factor calculations are non-linear.

The last part of the chain, the sensitivity of internal forces with respect to design variables $dN(x)/dx$, is performed in a separate step. This process is of interest because, unlike the reserve factors, all the internal forces are static responses, which can be described linearly and easily be computed in standard finite element programs. Therefore, it is possible to linearly combined the analysis results of a reduced set of characteristic loads to approximate the sensitivity due to the full set of original load cases.

5.2. Sensitivity of the internal loads with respect to the design variables

Fundamentally in finite element analysis, the sensitivity or the gradient of the static responses such as internal load, stress and strain energy is computed based on the displacement responses due to the applied external loads [11]. Since the internal load (N) is typically a function of displacements (D) and design variables (x), the gradient computation can be described by the calculus chain rule as

$$\frac{dN}{dx} = \frac{\partial N}{\partial x} + \frac{\partial N}{\partial D} \cdot \frac{\partial D}{\partial x} \quad (34)$$

Three partial derivative terms are required to compute the full sensitivity. The internal load appearing in the equation (34) is generally associated with an element stiffness matrix (C) and the displacement matrix as

$$N = CD \quad (35)$$

The formulation of C depends on the type of the structure being modelled. Since the explicit expression of internal load as a function of design variable is usually available, the terms $\partial N/\partial x$ and $\partial N/\partial D$ can normally be obtained by directly differentiating equation (35) as follows:

$$\frac{\partial N}{\partial x} = \frac{\partial C}{\partial x} D \quad (36)$$

$$\frac{\partial N}{\partial D} = C \quad (37)$$

Calculating a partial derivative of the displacement with respect to the design variable $\partial D/\partial x$ is slightly more complicated since there is no explicit equation for the displacement to differentiate.

The standard practice to obtain this term in finite element analysis is to differentiate the static equilibrium equation that describes the displacement of a structure when subjected to the applied external force (F) in the static equilibrium state by a global stiffness matrix (K) as

$$F = KD \quad (38)$$

Differentiating and rearranging this equation yields

$$K \frac{dD}{dx} = \frac{\partial F}{\partial x} - \frac{\partial K}{\partial x} D \quad (39)$$

The terms on the right hand side of equation (39) are referred as Pseudo-Loads. Usually, the applied external forces are independent of the design variables, so the term $\partial K/\partial x$ can be eliminated. Subsequently, rearrange this equation to obtain only $\partial D/\partial x$ on the left hand side as

$$\frac{\partial D}{\partial x} = -K^{-1} \frac{\partial K}{\partial x} D \quad (40)$$

Equation (39) can be solved directly to obtain the term $\partial D/\partial x$. Since the stiffness matrix (K) and the displacement matrix (D) are available from running the first finite element analysis, the change of the stiffness matrix with respect to the change in the design variable ($\partial K/\partial x$) can be computed, either by means of direct differentiation or via finite differencing as implemented in Nastran [12].

Substituting equation (36), (37) and (40) into equation (34), the sensitivity of the internal loads (N) with respect to the design variable (x) can be expressed in

terms of displacement (D) and stiffness matrices (C , K) as

$$\frac{dN}{dx} = \left(\frac{\partial C}{\partial x} - CK^{-1} \frac{\partial K}{\partial x} \right) D \quad (41)$$

5.3. Sensitivity of the internal loads with respect to the external applied forces

In order to validly apply the superposition of the characteristic loads, the linear relationship between the sensitivity of the internal loads (dN/dx) to the applied external loads should firstly be determined.

This task is relatively straightforward since the relationship between the sensitivity of the internal loads (N) as a function of displacement (D) has been established in equation (41). D is simply the displacement result, which can explicitly obtained from equation (38) as $K^{-1}F$. Substituting $K^{-1}F$ into equation (41) yields the equation of the sensitivity in terms of external forces as

$$\frac{dN}{dx} = \left(\frac{\partial C}{\partial x} - CK^{-1} \frac{\partial K}{\partial x} \right) K^{-1} F \quad (42)$$

Thus, the sensitivity of any internal load can be expressed by

$$\frac{dN}{dx} = MF \quad (43)$$

Equation (43) implies that the sensitivity and the external force matrix are linearly related by the M matrix. As it is already in the same format as equation (22), the superposition of k characteristic loads described in equation (27) is possible and can be given by

$$\frac{dN}{dx} = MPL_k^T U_k^T \quad (44)$$

Equation (44) describes the finite element process that analyses k characteristic loads (L) which has been rearranged by the permutation matrix (P) and subsequently multiplied by the coefficient matrix (U).

6. Cost analysis

There are many factors influencing the cost of the sensitivity analysis, but only three major terms are mentioned here: number of calculations, analysis time and the size of transferred data. Being able to quantify these terms allows the benefit of the load case reduction to be clearly understood.

6.1. Number of calculations

In linear calculation, the size of an output array normally implies the number of calculations needed to be

$$\begin{array}{c}
SE_1 \left\{ \begin{array}{c} dN_x/dx_1(1) \quad dN_x/dx_2(1) \quad \dots \quad dN_x/dx_n(1) \\ dN_y/dx_1(1) \quad dN_y/dx_2(1) \quad \dots \quad dN_y/dx_n(1) \\ \vdots \\ dP_x/dx_1(1) \quad dP_x/dx_2(1) \quad \dots \quad dP_x/dx_n(1) \end{array} \right. \\
SE_2 \left\{ \begin{array}{c} (2) \\ \vdots \\ (m) \end{array} \right. \\
SE_m \left\{ \begin{array}{c} (2) \\ \vdots \\ (m) \end{array} \right.
\end{array}$$

Figure 5: Example of the sensitivity matrix on a GFEM structure for a single load case

done. Figure 5 shows the typical array of a GFEM structure. The size of the sensitivity matrix as similar to this figure can roughly be estimated by

$$n_{total} = (n_N \times n_{SE}) \times n_{LC} \times n_x \quad (45)$$

such that n_N , n_{SE} , n_{LC} and n_x represents the number of internal loads, structural elements, load cases and design variables respectively.

The number of internal forces in a structural assembly depends on the type of structural element. For a general wing GFEM involving panel and stringer structural elements, there are 3 in-plane loads on each panel and 1 axial load on the stringer.

6.2. Analysis time

The analysis time is a very crucial factor in the optimisation process because the process may be repeated for many iterations before the optimal design is achieved. Reducing the sensitivity calculation time in each optimisation cycle contributes to the significant amount of time saving of the entire design process. Not only the design operation can be controlled within the limited time frame, but this also encourages the designers to experiment with more data i.e. load cases and design parameters.

Due to the linear relationship between time and number of calculations, the rate of increase of the analysis time can be determined from just a single finite element run of any number of load cases. This can be expressed by

$$\text{slope} = \frac{\text{time}_{nLC}}{n_{LC}} \quad (46)$$

6.3. Data size

In the aerospace industry, engineering programs are usually operated in the central computers via an internal network. Significantly large files may potentially cause some problems as follows:

- **Hard disk capacity:** Since the files have to be stored in the computer's hard disks, the full disc space prevents any new jobs to be invoked, especially when a company operates its software on the main computer servers.
- **Memory:** Printing and reading a very large file size consume considerable amount of memory and sometimes cause the system to crash.
- **Network transfer:** Transferring a set of large files through the system network could sometimes take a very long time to complete. It also consumes the network bandwidth and may limit the access speed of other users.

The size of files generated by a finite element program could be very large depending on which types of files are required. The files that contain finite element properties and relevant equations are normally constant regardless of the number of load cases. However, the size of output files depends greatly on the size of the output matrix, which is proportional to the number of load cases in static sensitivity analysis.

Similar to the analysis time, the data size is increased linearly as the number of load cases rises. The slope of the graph can be expressed as

$$\text{slope} = \frac{\text{size}_{nLC}}{n_{LC}} \quad (47)$$

7. Results & discussion

The experiment was conducted on the actual loading data on a wing and a typical wing GFEM so that the impact of the characteristic load reduction on the sensitivity analysis process can be clearly investigated.

The loading data was provided by the load department in Airbus (UK) as an array of load cases that were generated from a typical real aircraft database containing a mix of ground and flight load cases. It was similar to the one that would be used in the actual design process. The data was already desensitised by using a tool mapping the loads from a specific geometry to a simple box-beam representation.

7.1. Load case characterisation result

The size of the initial loading array is $[100 \times 162]$. The SVD was performed on this matrix to extract different sets of characteristic loads L_k such that $k = 1, \dots, 100$. It was computed using the predefined SVD function in MATLAB, which is extremely fast and efficient. For this specific scenario where there are less than 30×10^4 entries in a single matrix, the computational effort is considered insignificant. The characteristic loads were then linearly combined with the corresponding U_k matrix to approximate the A matrix. The SVD was also carried out for the balanced case.

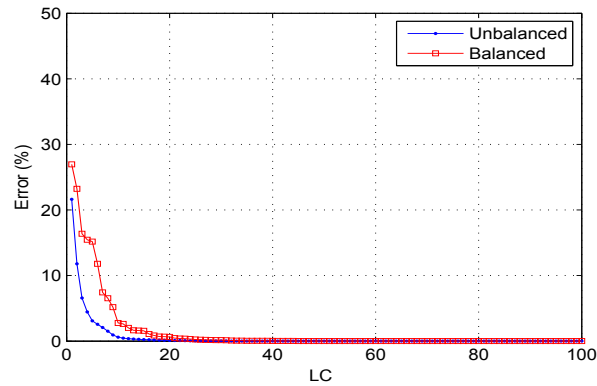


Figure 6: Original magnitudes of SMT loadings

Figure 6 demonstrates that the approximation error reduces when the number of characteristic loads increases. Error drops to almost zero at around 20 characteristic loads for the original data and 25 for the balanced load matrix.

7.2. Balancing

The result suggests that the balancing increases the approximation error calculated in the Frobenius norm. This happens probably because the Frobenius norm measures the magnitude of the overall matrix, and the magnitude of some loading values i.e. moments is too dominant in the unbalanced case. Since the SVD tends to capture higher magnitudes, only the moment values were well captured.

However in the analysis stage, other values such as shear forces are also important. More detailed investigation, for example error of each loading component at the individual station, is required before the decision between the original and balancing approaches can be made.

Further study focusing on the effect of balancing has been conducted. Figure 7 shows that the relative errors at each wing station, which were calculated from

the sum of the relative error at each column of the matrix divided by the number of columns. The balancing improves the load cases reconstruction quality in general.

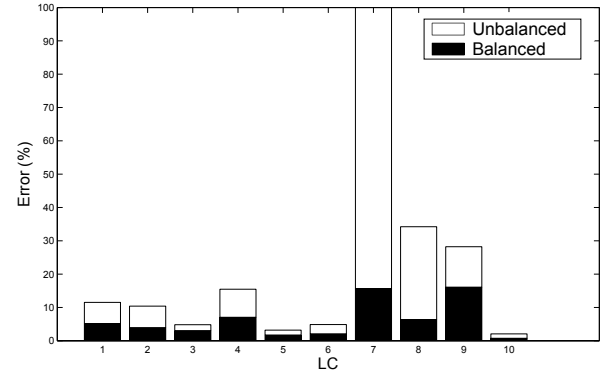


Figure 7: Average error per station of the 10 lowest magnitude cases

A plot of the shear force in the chord-wise direction of the wing around the wing root stations is also shown in figure 8. It can clearly be seen that balancing improves the mapping quality.

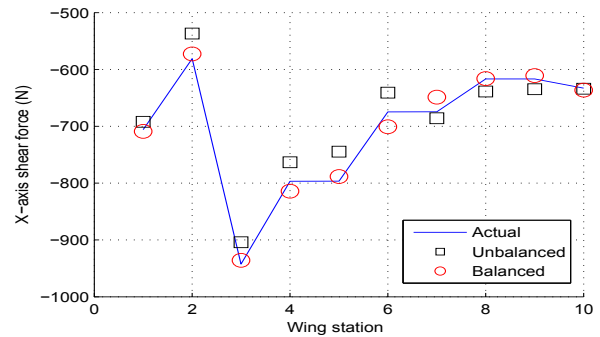


Figure 8: The reconstruction results of the x-axis shear forces around the wing root stations of a low magnitude load case

The future work, the error in other norms (e.g. infinity norm, 2-norm) can be investigated to minimise the error in the quantities of interest.

7.3. Analysis of characteristic loads results

Different sets of characteristic load cases, ranged from 1 to 100, were carried over to analyse in the wing GFEM. The 100 actual load cases were also analysed to be used as the based reference in the error calculation. Only the results from the balancing method will be shown here.

From several input files generated, the main file that is of concern is the one that contains the loading information. The characteristic loadings were firstly

transformed into the GFEM geometry using the 6DOF shear moment diagram and then inserted into the input file. The analysis was performed in the MD Nastran 2011, which has been installed on a Windows 7 Enterprise 64bit machine with Intel Xeon 3.10 GHz processor and 16 GB memory.

Sensitivity results were superimposed from a set of characteristic load cases. Figure 9 shows the approximation error of the analysis and the external loads. The superposition error is eliminated when the number of characteristic loads is around 25. The trend of the results complies with equation (32), which suggests that the superposition should possess a similar trend with the error at the external loads level. The maximum errors in

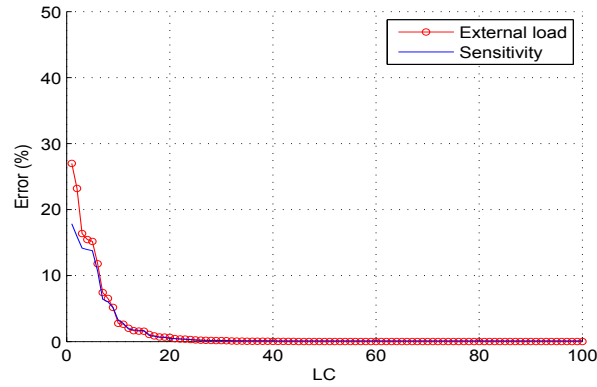


Figure 9: The comparison between the errors from the external and the analysis level

each design variable column are also less than 5% relative its norm. Hence, 25 characteristic loads should be sufficient to approximate the sensitivity values.

7.4. Cost analysis

Three types of information were studied including data size, analysis time against the error from each characteristic load run against the approximation error. The analysis time and the size of output files were obtained from equation (46) and (47) respectively. The differences between the actual sensitivity matrix and the approximated were normalised by the Frobenius norm and compared with the Frobenius norm of the actual matrix to obtain the approximation errors. These 3 types of information were plotted together as shown in figure 10.

The relationship between the approximation error and computation effort is inversely proportional. The error approaches zero when increasing the number of characteristic loads to around 25 whereas the file size and the analysis time continue to increase. Instead of running the full 100 load cases, the analysis of only 25 characteristic loads would provide computational savings of around 3 times.

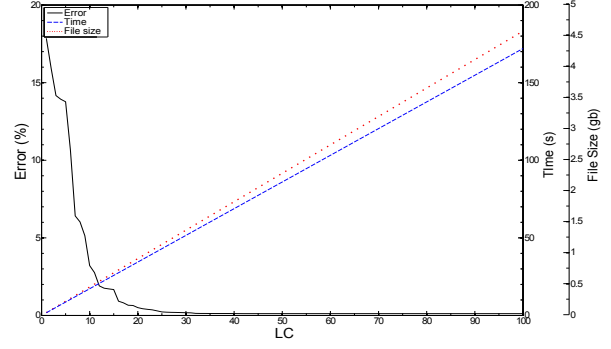


Figure 10: Approximation error, analysis time and file size at different numbers of characteristic loads of a single structural element

It is important to remark that only a structural element consisting of 1 stringer and 2 panels with around 60 design variables was run in this experiment. There are approximately, according to equation (47), around $7 \times 60 = 420$ entries in the sensitivity matrix generated.

In contrast, a typical industrial wing GFEM model is an assembly of approximately 500 structural elements. The number of rows and columns of the matrix will increase by a factor of 500^2 . Hence for a full wing GFEM, the size of the sensitivity matrix would be around $420 \times 500^2 = 10.5 \times 10^6$ per single load case. Eliminating 1 load case significantly reduces the computational effort.

Table 1 demonstrates the potential cost reduction that can be achieved when using 25 characteristic loads instead of 100 load cases in the sensitivity calculation of a full wing GFEM.

Table 1: Comparison of computational effort between the full and characteristic load cases analysis (estimated for a full wing GFEM)

Analysis type	Time (hr)	Size (TB)	Error(%)
100 full LCs	1194	114	-
25 characteristic LCs	298	28	0.22

8. Conclusion

This paper presents a new approach to characterise aircraft loading and its application in the gradient based optimisation process. A set of smaller important loadings called characteristic loads can be identified from the full loadings by performing SVD. The linear relationship between the sensitivity matrix of the internal loads with respect to the design variables and the external loading matrix in a finite element model indicates that it is possible to analyse the characteristic loads and then use linear combination of the reduced rank SVD approximation to create the full set of sensitivity results.

Using the Frobenius norm as the error indicator provides a good and efficient way of decision making of how many characteristic loads are required to create well approximated results. Detailed error investigation may be required, especially in an application where the accuracy is crucial. More characteristic loads can be included to improve the accuracy but will also increase the analysis time. The balancing method, which is an approach to adjusting the magnitude of the loading matrix before performing the SVD, will also improve the accuracy.

In the experiment, different sets of characteristic load cases were extracted from 100 load cases. The result suggests that only a quarter of the full load cases is sufficient to create the original results. The set of characteristic loads were then analysed in the wing GFEM to create the approximated sensitivity results. By using only a quarter of the original load cases in the analysis, significant computational reduction can be achieved. For a typical GFEM wing, it effectively eliminates around 10.5×10^6 calculations for each load case removed. The study also concludes that the analysis time and the file size were reduced by 75%.

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